UPDATE

International Mathematics Olympiad

More than a century has passed since the first Eötvös competition was organised in Hungary in 1894. This competition can be called the forerunner of the International Mathematical Olympiad (IMO). For record, the IMO began in Romania in 1959 with Bulgaria, Czechoslovakia, East Germany, Hungary, Poland, Romania and USSR as participants and has grown enormously over the years.

India sent her team to IMO for the first time in 1989 to the 30th IMO. Since then it has been sending a team to IMO every year. India hosted the IMO in Mumbai from 5th July to 17th July, 1996. This was the 37th IMO. A staggering number of 75 countries participated in the prestigious event. Earlier Rina Panigrahy won a gold medal for India in 1990 at 31st IMO in Beijing, Peoples Republic of China.

How Indian Team is Selected

Indian team has been consistently faring well at the IMOs. The selection of the team for IMO is a three-stage process. The first stage consists of holding Regional Mathematical Olympiads (RMOs). For this purpose, the whole country is divided into sixteen regions. RMOs are usually conducted in each region sometime between October and December. A maximum of about twenty-five students are picked from each region for the second stage. About four hundred or so students selected from all over the country write the Indian National Mathematical Olympiad (INMO), which is held on the first Sunday of February every year. On the basis of the INMO about thirty students are selected for further training in problem solving from the finest minds of India at institutions of excellence such as Tata Institute of Fundamental Research (TIFR), Mumbai; Indian Institute of Science (IISc), Bangalore and the like. Participants are given special training in the month of May and a team of six best students is chosen to represent India at the IMO and win laurels for the country.

National Board of Higher Mathematics (NBHM) organises the National Mathematical Olympiad contests on behalf of the Government of India. NBHM is working under the Department of Atomic Energy.

Degree of Co-relation between Performance at IMO/RMO/INMO and at IIT-JEE

Indeed, it is a fact that high degree of co-relation exists between performance at RMO/IMO/INMO and that at IIT-JEE. Almost all the RMO winners secure very high ranks at IIT-JEE. IMO team members always secure top ranks at IIT-JEE. To name the same, Rina Panigrahy IMO-90 gold medallist had All-India Rank (AIR) 1 at IIT-JEE 1991, Subhash Abhijit Khot, top scorer at IMO-95 had AIR 1 at IIT-JEE 1995, and Ajay C. Ramdoss, gold medallist at IMO-96 secured AIR 6. And the story continues.....

The INMO awardees are eligible for attractive scholarships during their studies beyond 10 + 2 stage, provided they study Mathematics as the main subject in their under-graduate (B.Sc.) Course. The scholarship at present is Rs. 700 p.m. The awards are in addition bestowed on them Rs. 1000 per annum as books grant.

How IMO Problems are Selected ?

A unique feature of IMO contest is that participating countries themselves propose the problems which vary from five to six per country. It is the job of the 'Problems Committee' of the host country to shortlist about thirty problems from among those that are received from participating countries. Each country which accepts the invitation to participate in the IMO sends a delegation consisting of the leader of the delegation, his deputy and six contestants. The IMO is open only to secondary school students or to pupils of schools on an equal level.

The leader of delegation and his deputy should be mathematicians or mathematics teachers. They must be able to express themselves exactly and clearly on mathematical and technical aspects of the contest in at least one of the official languages of the IMO, *viz.*, English, French, German or Russian.

The problems proposed by the participating countries to the host country along with solutions, formulated in one of the official languages of the Olympiad, go to the Jury. It is expected that problems should come from various areas of mathematics, such as included in math curricula at high schools. The solutions of these problems should, however, require exceptional mathematical ability and excellent mathematical knowledge on part of the contestants. It is assumed that the problems will be original and not yet published anywhere. The host country—the country that organises the Olympiad in a particular year—does not submit any problems for the contest.

The Jury takes decisions on questions that are related to the preparation and realisation of the contest paper and to the evaluation of its result in accordance with the provisions of these regulations.

Plan of the Examination : Then from the preliminary broader selection of problems prepared by the 'Problem Selection Committee', 6 problems are selected for the contest. The sequence of problems for the contest and their division under the two days of the event are also finalised by Jury. The contest is spread over two days, each day having three problems to be solved in 4½ hours duration.

Awarding of Prizes : Each participant receives a Diploma certifying to his participation in the Olympiad. The most successful contestants will be awarded 1st, 2nd and 3rd prizes. Special awards may be presented for outstanding original solutions.

In the grading of papers there are extra marks for elegant solutions and/or non-trivial generalisations with proof. Although generalisations are part and parcel of mathematical creativity and elegant solutions are much more satisfactory and transparent than non-elegant ones, finding them usually takes time. But if

there is time, contestants are advised to strive for refinements, since elegance is almost always a sign of real understanding and generalisation a sign of creativity.

The Syllabus of the IMO : Here lies the rub. There is no fixed syllabus for the IMOs or for that matter RMOs/INMO. What is needed is a sound understanding of fundamentals at the Secondary School level. Only the questions are of non-routine type and profound understanding and deep insight is required for the solution of the problems.

However, from the previous years IMOs and for the benefit of young mathematical minds we give an idea of what normally is asked in IMOs. Following areas emerge after a careful examination of previous Olympiad contests.

1. **Algebra :** Arithmetic Mean, Geometric Mean Theorem, Cauchy-Schwarz Inequality, Tchebycheff's Inequality, Symmetric Functions of the Roots of an Equation, Theory of Equations.

2. **Number Theory :** Principle of Mathematical Induction, Tests of Divisibility, Greatest Common Divisor, Euclid's Algorithm, Unique Factorisation Theorem, Congruency, Theorems of Fermat and Wilson and their applications.

3. **Plane and Solid Geometry :** Circles, Ratio and Proportion, Ceva's Theorem, Henalus' Theorem, Ptolmey Theorem, Lehmus-Steiner Theorem, Properties of the Nine-point Circle, Simson's Line, Centres of Similitude of Two Circles, Tetrahedron/Cube.

4. **Combinatorics :** Recurrence Relation, Derangements, Principle of Counting, Pigeon-hole Principle, Inclusion Exclusion problem.

5. **Geometric and Algebraic Inequalities :** Beautiful applications of A.M.-G.M. Theorem.

India in IMO 2004

The 45th International Mathematical Olympiad (IMO-2004) was held at Athens, Greece. (6th July-18th July, 2004). A total of 486 students from as many as 85 countries participated in the event.

Two participants from China, one from Hungary and one from Canada got a perfect score of 42. For Gold, Silver and Bronze medals the respective cut-offs were pegged at 32, 24 and 16.

India did well by securing 4 silver medals and 2 bronze medals won by six of its participants.

	Candidate Name	*Place*	*Score*	*Medal Received*
1.	Abhishek Dang	Pune	19	Bronze
2.	Anand Deopurkar	Pune	30	Silver
3.	Anupam Prakash	Ranchi	23	Bronze
4.	Kshipra Bhavalkar	Pune	25	Silver
5.	Rohit Joshi	Pune	24	Silver
6.	Vipul Naik	Delhi	30	Silver

Books Recommended for the IMO

1. **Algebra**

- J.V. Uspenspey : *Theory of Equations*, Mc-Graw Hill, 1945.
- C.V. Purell and A. Robson : *Advanced Algebra*, Vols. I, II, III, Bell, London 1964.
- G. Chrystal : *Algebra*, Vol. I, II, Chelsea, N.Y., 1952.
- A. Mostowski and H. Stork : *Introduction to Higher Algebra*, Pergamon, Oxford, 1964.

2. **Number Theory**

- H. Rademacher : *Lectures on Elementary Number Theory*, Blaisdell, N.Y. 1954
- I. Viven and Zuckesman : *An Introduction to the Theory of Numbers*, Wetey, N.Y., 1960.

3. **Plane and Solid Geometry**

- Bell : *An Elementary Treatise on Coordinate Geometry of Three Dimensions*, Macmillan, London, 1912.
- A. Presden : *Solid Analytical Geometry and Determinants*, Wiley, N.Y., 1930
- H. Eves : *A survey of Geometry*, Vols. I and II, Allyn and Bacon, Boston, 1963.
- G. Salmon : *A Treatise on Conic Sections*, Chelsea, N.Y., 1954.
- H.S.M. Coxeter : *Introduction to Geometry*, Wiley, N.Y., 1969.

4. **Combinatorics**

- C. Berge : *Principles of Combinatorics*, Academic Press, N.Y., 1971.
- A. Tucker : *Applied Combinatorics*, Wiley, N.Y., 1980.
- W.A. Whitworth : *Choice and Chance*, Hafner, N.Y., 1948.
- E. Borel : *Elements of the Theory of Probability*, Prentice Hall, N.Y., 1950.
- W. Burnside : *Theory of Probability*, Dover, N.Y., 1959.

5. **Geometric and Algebraic Inequalities**

- Hardy, Littlewood and Pulya : Inequalities, Cambridge Universities Press, Cambridge, 1934.
- D.S. Mitrinovic : *Elementary Inequalities*, Nordhoff, Groningen, 1964.

Our aim has been to help students satisfy his curiosity by giving him a thorough detail of the celebrated Olympiad contest and mathematicians the world over are convinced of the educational impact of such problems in stimulating thinking of mathematically gifted bright students. What better way to close the article than to remember what great problem solver G. Szegös has to say on spirit of problem solving : "We should not forget that the solution to any worthwhile problem, very rarely, comes to us easily and without hard work; it is rather the result of intellectual effort of days or weeks or months. Why should the young mind be witting to make this supreme effort ? The explanation is probably the instinctive preference of certain values, that is, the attitude which rates intellectual efforts and spiritual achievement higher than the material advantage."

International Olympiad Problems *with Solutions*

The International Mathematical Olympiad has been held annually since 1959. The problems are solvable by methods accessible to Secondary School Students in most nations, but insight and ingenuity are often required.

Problems

1. Show that the curve $y=\dfrac{x+1}{x^2+1}$ has three points of inflection lying in a straight line.

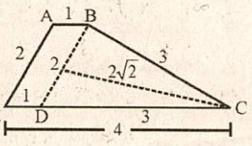
2. Find the least whole number which begins with the digit 1 and increases 3 times when this digit is carried to the end of the number. Find all the numbers possessing this property.

3. For which real numbers c is $\dfrac{\left(e^x+e^{-x}\right)}{2}\le e^{cx^2}$ for all real x ?

4. Compute $\displaystyle\int_0^{\pi/2}\frac{dx}{1+(\tan x)^{\sqrt{2}}}$.

5. The function $y=f(x)$ is represented parametrically :
$$\begin{cases} x=\varphi(t)=t^5-5t^3-20t+7 \\ y=\psi(t)=4t^3-3t^2-18t+3\,(-2<t<2) \end{cases}$$
Find the extrema of this function.

6. The sides of a trapezoid are one, two, three and four centimetres long.



   (a) What is the area of the trapezoid ?

   (b) Show that the answer is unique.

7. Prove that there are infinitely many positive integers n with the property that if p is a prime divisor of n^2+3, then p is also a divisor of k^2+3 for some integer k with $k^2<n$.

8. Compute the infinite sum A.
$$A=\frac{1}{2}-\frac{2}{4}+\frac{3}{8}-\frac{4}{16}+\frac{5}{32}-\ldots..+\frac{n}{2^n}(-1)^{n+1}+\ldots.$$

9. Show that the square roots of two successive natural numbers greater than N^2 differ by less than $1/(2N)$.

10. Prove that for $0\le p\le 1$ and for any positive a and b the inequality $(a+b)^p\le a^p+b^p$ is valid.

Solutions

Q.1. Find the derivatives :
$$y'=\frac{-x^2-2x+1}{\left(x^2+1\right)^2},\quad y''=\frac{2x^3+6x^2-6x-2}{\left(x^2+1\right)^3}.$$

Also $y''=0$ at three points, which are the roots of the equation

$x^3+3x^2-3x-1=0$

$\Rightarrow\quad x_1=-2-\sqrt{3},\ x_2=-2+\sqrt{3},\ x_3=1$

Table of signs of y'' is given below :

x	$-\infty<x<-2-\sqrt{3}$	$-2-\sqrt{3}<x<-2+\sqrt{3}$	$-2+\sqrt{3}<x<1$	$1<x<\infty$
Sign of y''	$-$	$+$	$-$	$+$
Conclusion	Convexity	Concavity	Convexity	Concavity

Hence, $\left(-2-\sqrt{3},-\dfrac{\sqrt{3}-1}{4}\right),\left(-2+\sqrt{3},\dfrac{1+\sqrt{3}}{4}\right),(1,1)$ are points of inflection.

Q.2. Let N be the (m-digit) number that is obtained when the initial digit 1 deleted in the sought-for number.

$\therefore$ By hypothesis, $(1.10^m+X).3=10\,N+1$

$\Rightarrow\quad X=\dfrac{3.10^m-1}{7}$

We can easily find the number X. Here we shall apply the process of long division of the number $3.10^m=30000\ldots$ by 7 till 1 is obtained in the remainder.

Consider
$$\begin{array}{r} 42857 \\ 7\overline{)3000\ldots0} \\ \underline{-28} \\ 20 \\ \underline{-14} \\ 60 \\ \underline{-56} \\ 40 \\ \underline{-35} \\ 50 \\ \underline{-49} \\ 1 \end{array}$$

$\therefore$ Least possible value of the number N is 42 857 and the least possible value of the sought-for number is 142 857.

After the first digit 1 is obtained, the process of long division could be continued till the next digit 1 is obtained and so on.

Thus, $\underbrace{\underbrace{142\,857}\ \underbrace{142\,857}\ldots\underbrace{142\,857}}_{k\text{ times}}$

which satisfies the condition of the problem.

Q.3. This inequality holds if and only if $c\ge 1/2$.

Now for $c\ge 1/2$, since $(2n)!\ge 2^n n!$ for $n=0,1,\ldots$

$$\frac{e^x+e^{-x}}{2}=\sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!}\le\sum_{n=0}^{\infty}\frac{x^{2n}}{2^n n!}=e^{x^2/2}\le e^{cx^2}\quad\text{for all } x.$$

Also, conversely, when the inequality holds for all x, then

$$0\le\lim_{x\to 0}\frac{e^{cx^2}-\frac{1}{2}\left(e^x+e^{-x}\right)}{x^2}$$

$$= \lim_{x\to 0} \frac{\left(1+cx^2+\ldots\right)-\left(1+\frac{1}{2}x^2+\ldots\right)}{x^2} = c - \frac{1}{2}$$

and so $c \geq 1/2$.

Q.4. Let $I = \int_0^{\pi/2} \frac{dx}{1+(\tan x)^{\sqrt{2}}}$. Put $x = \left(\frac{\pi}{2}\right) - u$ and $r = \sqrt{2}$

$$\therefore \quad I = \int_{\pi/2}^{0} \frac{-du}{1+\cot^r u} = \int_0^{\pi/2} \frac{\tan^r u\, du}{\tan^r u + 1}$$

$$\therefore \quad 2I = \int_0^{\pi/2} \frac{1+\tan^r x}{1+\tan^r x}dx = \int_0^{\pi/2} dx = \frac{\pi}{2} \quad \Rightarrow \quad I = \frac{\pi}{4}.$$

Q.5. We have, $\varphi'(t) = 5t^4 - 15t^2 - 20$.

In the interval $(-2, 2)$ $\varphi'(t) \neq 0$.

Then $\psi'(t) = 0$ gives $12t^2 - 6t - 18 = 0$

$\Rightarrow$ $t_1 = -1$ and $t_2 = 3/2$.

These roots are interior points of the considered interval of variation of the parameter t.

Also, $\psi''(t) = 24t - 6$; $\psi''(-1) = -30 < 0$, $\psi''(3/2) = 30 > 0$.

Consequently, the function $y = f(x)$ has a maximum $y = 14$ at $t = -1$, (*i.e.*, at $x = 31$) and a minimum $y = -17.25$ at $t = 3/2$, (*i.e.*, at $x = -1033/32$).

Q.6. We have from the figure, $\overline{AB} \,||\, \overline{ED}$ and $\overline{AE} \,||\, \overline{BD}$.

Also altitude from C to $\overline{BD} = 2\sqrt{2}$.

Since the area of ΔBCD is the same regardless of which leg is chosen as the base, if x is the length of the altitude from B to $\overline{DC}$, then

$$\frac{1}{2}(2)2\sqrt{2} = \frac{1}{2}(3)x \quad \Rightarrow \quad x = \frac{4\sqrt{2}}{3}$$

$$\therefore \quad \text{Area of the trapezoid} = \frac{1}{2}(4+1)\left(\frac{4\sqrt{2}}{3}\right) = \frac{10\sqrt{2}}{3}.$$

All other possibilities for such a trapezoid would violate the triangle inequality.

Q.7. Since $m \in I^+$ m, ranges through all non-negative integers,

$$n = (m^2 + m + 2)(m^2 + m + 3) + 3$$

takes on an infinite set of positive integral values.

Put $f(x) = x^2 + 3$. Examination of $\{f(m)\} = 3, 4, 7, 12, 17, 28, 39, 52, 67, 84, \ldots\ldots$ leads one to conjecture that

$$f(x)f(x + 1) = f[x(x + 1) + 3] = f(x^2 + x + 3).$$

Using this property and the above relation between m and n,

$$f(n) = f(m^2 + m + 2)f(m^2 + m + 3)$$
$$= f(m^2 + m + 2)f(m)f(m + 1).$$

Thus, $p\,|\,f(n)$ with p prime implies that $p\,|\,f(k)$ with k equal to m, $m + 1$, or $m^2 + m + 2$.

Since each of these possibilities for k satisfies $k^2 < n$.

Hence, the result.

Q.8. We have, $A = \frac{1}{2} - \frac{2}{4} + \frac{3}{8} - \ldots$; $2A = 1 - \frac{2}{2} + \frac{3}{4} \ldots$ adding terms with the same denominator produces

$$3A = 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} \ldots = \frac{2}{3}. \text{ So, } A = \frac{2}{9}.$$

Q.9. Apply the Lagrange formula to the function $f(x) = \sqrt{x}$ on the interval $[n, n + 1]$

$$f(n+1) - f(n) = \sqrt{n+1} - \sqrt{n} = \frac{1}{2\sqrt{\xi}}, \text{ where } n < \xi < n+1.$$

If $n > N^2$, then $\xi > N^2$, hence $\frac{1}{2\left(\sqrt{\xi}\right)} < \frac{1}{(2N)}$

$$\Rightarrow \quad \sqrt{n+1} - \sqrt{n} < 1/(2N).$$

Q.10. By dividing both sides of the inequality by b^p, we get

$$\left(\frac{a}{b}+1\right)^p \leq \left(\frac{a}{b}\right)^p + 1 \text{ or } (1+x)^p \leq 1 + x^p \qquad (*)$$

where $x = \frac{a}{b}$.

Let us show that the inequality (*) holds true at any positive x. Introduce the function

$$f(x) = 1 + x^p - (1 + x)^p;\ x \geq 0.$$

The derivative of this function

$$f'(x) = px^{p-1} - p(1+x)^{p-1} = p\left[\frac{1}{x^{1-p}} - \frac{1}{(1+x)^{1-p}}\right]$$

is positive everywhere, since, by hypothesis, $1 - p \geq 0$ and $x > 0$. Hence, the function increases in the half-open interval $[0, \infty)$, *i.e.*, $f(x) = 1 + x^p - (1 + x)^p > f(0) = 0$, whence

$$1 + x^p > (1 + x)^p,$$

which completes the proof. If we put $p = 1/n$, then we obtain

$$\sqrt[n]{a+b} \leq \sqrt[n]{a} + \sqrt[n]{b} \quad (n \geq 1).$$

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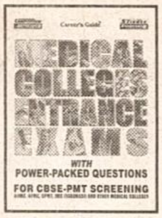

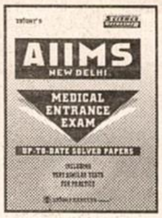

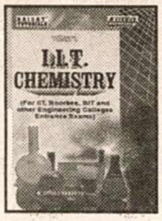

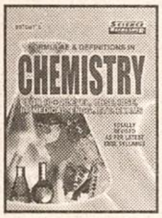

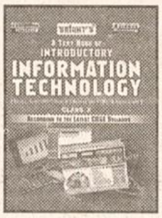

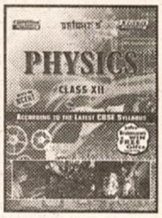

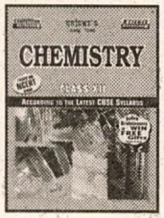

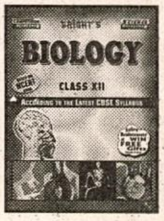

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Brain Teaser

Q.1. *ABCD is a rhombus. Its diagonals AC and BD intersect at the point M and satisfy BD = 2AC. Its points D and M represent the complex numbers 1 + i and 2 − i respectively. Find the complex number represented by A.*

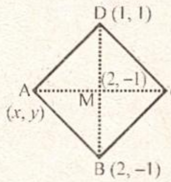
Sol. Let A be (x, y).

It is given that BD = 2AC $\Rightarrow$ MD = 2AM

Also DM is perpendicular to AM

$\Rightarrow \quad (1-2)^2 + (1+1)^2 = 4[(x-2)^2 + (y+1)^2] \qquad ...(i)$

and $\dfrac{y+1}{x-2}\cdot\dfrac{1+1}{1-2} = -1 \Rightarrow 2(y+1) = x-2$



with $x - 2 = 2(y+1)$, (i) gives

$$(y+1)^2 = \frac{1}{4} \Rightarrow y = -\frac{1}{2}, \frac{3}{-2} \Rightarrow x = 3, 1$$

$\Rightarrow$ A represents $z = 3 - \dfrac{i}{2}$, or $1 - \dfrac{3i}{2}$.

Q.2. *Let* $I_m = \displaystyle\int_0^\pi \frac{1-\cos mx}{1-\cos x}\,dx$. *Use mathematical induction to prove that* $I_m = m\pi$, $m = 0, 1, 2 \ldots\ldots$.

Sol. $\displaystyle I_1 = \int_0^\pi \frac{1-\cos x}{1-\cos x}\,dx = \pi$

$$I_2 = \int_0^\pi \frac{1-\cos 2x}{1-\cos x}\,dx = 2\int_0^\pi (1+\cos 2x)\,dx = 2\pi.$$

P(1) and P(2) are true. Suppose P$(k-1)$ and P(k) are true.

Now $2I_k - I_{k-1} - I_{k+1}$

$$= \int_0^\pi \frac{2(1-\cos kx) - [1-\cos(k-1)x] - [1-\cos(k+1)x]\,dx}{(1-\cos x)}$$

$$= \int_0^\pi \frac{[\cos(k-1)x - \cos kx] + [\cos(k+1)x - \cos kx]\,dx}{2\sin^2\frac{x}{2}}$$

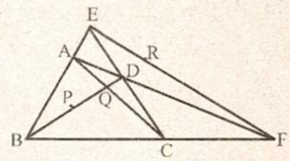
$$= \int_0^\pi \frac{2\sin\frac{x}{2}\left[\sin\left(\frac{2k-1}{2}\right)x - \sin\left(\frac{2k+1}{2}\right)x\right]dx}{2\sin^2\frac{x}{2}}$$

$$= \int_0^\pi \frac{2\sin\frac{x}{2}\times 2\sin\left(-\frac{x}{2}\right)\cos kx}{2\sin^2\frac{x}{2}}\,dx = \int_0^\pi \cos kx\,dx = 0$$

$\Rightarrow \quad I_{k+1} = 2I_k - I_{k-1} = (k+1)\pi \Rightarrow$ P$(k+1)$ is true.

Q.3. *Show that the mid-points of the three diagonals of a complete quadrilateral are collinear.*

Sol. A complete quadrilateral is a figure made by four straight lines, no three of which are concurrent. Let ABCD be a quadrilateral and let P, Q and R be the respective mid-points of the diagonals BD, CA and EF. Let $\vec{a}, \vec{b}, \vec{c}, \vec{d}, \vec{e}, \vec{f}, \vec{p}, \vec{q}$ and $\vec{r}$ be the position vectors of A, B, C, D, E, F, P, Q and R relative to some origin. Then



$$\vec{p} = \frac{\vec{b}+\vec{d}}{2},\ \vec{q} = \frac{\vec{c}+\vec{a}}{2} \text{ and } \vec{r} = \frac{\vec{e}+\vec{f}}{2}.$$

Since E lies on the two lines BC and AD and F lies on the two lines BA and CD respectively, we have

$$\vec{e} = (1-s)\vec{b} + s\vec{c} \qquad \vec{e} = (1-t)\vec{a} + t\vec{d}$$

$$\vec{f} = (1-u)\vec{b} + u\vec{a} \qquad \vec{f} = (1-v)\vec{c} + v\vec{d}$$

where s, t, u and v are scalars. Adding all these relations, we get

$$(1-t+u)\vec{a} + (1-s-u)\vec{b} + (1+s-v)\vec{c} + (t+v)\vec{d} - 2(e+f) = 0$$

Setting $1 - t + u = 1 + s - v$ and $2 - s - u = t + v$

So that, $s + t = u + v = 1$, we get

$$(2-s-u)(\vec{b}+\vec{d}) + (1-t+u)(\vec{c}+\vec{a}) - 2(\vec{e}+\vec{f}) = 0$$

$\Rightarrow \quad (2-s-u)\vec{p} + (1-t+u)\vec{q} - 2\vec{r} = 0$

This shows that the points P, Q and R are collinear, since the sum $(2 - s - u) + 1 - t + u - 2 = 0$.

Q.4. *Show that all the values of a for which the inequality*

$$\frac{1}{\sqrt{a}}\int_1^a \left(\frac{3}{2}\sqrt{x} + 1 - \frac{1}{\sqrt{x}}\right)dx < 4$$

is satisfied, lie in the interval (0, 4).

Sol. $\displaystyle \frac{1}{\sqrt{a}}\int_1^a \left(\frac{3}{2}\sqrt{x} + 1 - \frac{1}{\sqrt{x}}\right)dx$

$$= \frac{1}{\sqrt{a}}\left(x^{\frac{3}{2}} + x - 2x^{\frac{1}{2}}\right)\Bigg|_1^a = a + \sqrt{a} - 2$$

$\therefore \quad a + \sqrt{a} - 2 < 4 \Rightarrow a + \sqrt{a} < 6 \Rightarrow \sqrt{a} < 6 - a$

Hence $a \le 6$. Squaring the inequality, we get

$$a < 36 + a^2 - 12a \Rightarrow 0 < a^2 - 13a + 36$$

$\Rightarrow \quad 0 < (a-9)(a-4)$

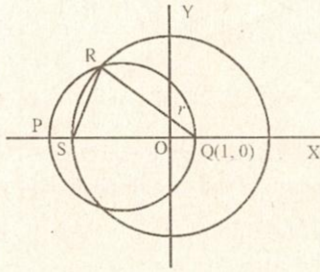
Hence $a > 9$ or $a < 4$. Since for $a < 0$, $\sqrt{a}$ is meaningless, we have

$$a \in [(9, \infty) \cup (0, 4)] \cap (0, 6] = (0, 4).$$

Q.5. *The circle $x^2 + y^2 = 1$ cuts the x-axis at P and Q. Another circle with centre at Q and variable radius intersects the first circle at R above the x-axis and the line segment PQ at S. Find the maximum area of the triangle QSR.*

Sol. Suppose the circle, with centre at Q(1, 0) has radius r. Since this circle has to meet the first circle, $0 < r < 2$. Equation of the circle with centre at Q(1, 0) and radius r is

$$(x - 1)^2 + y^2 = r^2 \qquad ...(i)$$



Let us find the coordinates of point R. For this we solve $x^2 + y^2 = 1$ and eqn. (i) simultaneously.

Subtracting $x^2 + y^2 = 1$ from eqn. (i), we get

$$(x - 1)^2 - x^2 = r^2 - 1$$

$$\Rightarrow \quad 1 - 2x = r^2 - 1$$

or $\quad x = \dfrac{2 - r^2}{2}$

Putting this in $x^2 + y^2 = 1$, we get

$$y^2 = 1 - \frac{1}{4}\left(2 - r^2\right)^2 = \frac{4 - \left(r^4 - 4r^2 + 4\right)}{4}$$

$$= \frac{4r^2 - r^4}{4} = \frac{r^2\left(4 - r^2\right)}{4} \quad \text{or} \quad y = \frac{r\sqrt{4 - r^2}}{2}.$$

Also, note that coordinates of S are $(1 - r, 0)$, therefore,

$$SQ = 1 - (1 - r) = r$$

Let A denote the area of ΔQSR, then

$$A = \frac{1}{2} r\left[\frac{r\sqrt{4 - r^2}}{2}\right] = \frac{1}{4} r^2\sqrt{4 - r^2}$$

Note that A will be maximum if and only if $f(r) = r^4(4 - r^2)$ is maximum.

We have $f'(r) = 16r^3 - 6r^5 = 2r^3(8 - 3r^2)$

Setting $f'(r) = 0$, we have $2r^3(8 - 3r^2) = 0$. As $0 < r < 2$,

we get $r = \dfrac{2\sqrt{2}}{\sqrt{3}}$. Also, $f''(r) = 48r^2 - 30r^4$

We have, $f''\left(\dfrac{2\sqrt{2}}{\sqrt{3}}\right) = 48\left(\dfrac{4 \times 2}{3}\right) - 30\left(\dfrac{4 \times 2}{3}\right)^2$

$$= 128 - \frac{640}{3} = -\frac{256}{3} < 0$$

Thus, $f(r)$ maximum when $r = \dfrac{2\sqrt{2}}{\sqrt{3}}$.

Hence, maximum value of A is

$$\frac{1}{4}\left(\frac{8}{3}\right)\sqrt{4 - \frac{8}{3}} = \frac{4}{3\sqrt{3}} = \frac{4\sqrt{3}}{9}.$$

Q.6. *By considering colouring k out of n distinct items using one of the two colours for each item, show that*

$$\left({}^nC_0\right)\left({}^nC_k\right) + \left({}^nC_1\right)\left({}^{n-1}C_{k-1}\right) + ... + \left({}^nC_k\right)\left({}^{n-k}C_0\right) = 2^k\left({}^nC_k\right).$$

Sol. Let the two colours be C_1 and C_2. For the left hand side, to choose and colour k items out of n, we may proceed as follows :

(a) We choose $i (0 \le i \le k)$ items out of n and colour them C_1.

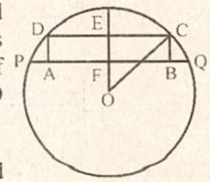
(b) Then we choose the $k - i$ items out of $n - i$ items and colour them C_2. The number of ways of choosing i items out of n is ${}^nC_i$, and the number of ways of choosing $k - i$ items out of $n - i$ is ${}^{n-i}C_{k-i}$. Thus, the number of ways of colouring k items out of n distinct items with one of the two colours is

$$\sum_{i=0}^{k}\left({}^nC_i\right)\left({}^{n-i}C_{k-i}\right) = \left({}^nC_0\right)\left({}^nC_k\right) + \left({}^nC_1\right)\left({}^{n-1}C_{k-1}\right) + ... + \left({}^nC_k\right)\left({}^{n-k}C_0\right)$$

For the right hand side, we choose k items out of n. This can be done in ${}^nC_k$ ways. For each item, we have two choices to colour C_1 or C_2. Therefore, the number of ways of choosing and colouring k items with one of the two colours is $({}^nC_k)2^k$.

Hence, $({}^nC_0)({}^nC_k) + ({}^nC_1)({}^{n-1}C_{k-1}) + ({}^nC_2) \times ({}^{n-2}C_{k-2}) + ... + ({}^nC_k)({}^{n-k}C_0) = ({}^nC_k)2^k$.

Q.7. *A chord of length 2L divides a circular area of radius R into two segments. Find the sides of the rectangle with the largest area that can be inscribed in the smaller segment.*



Sol. Let ABCD be the rectangle inscribed in the smaller segment of the circle of radius R with centre O. Let E be the mid-point of DC and $\angle$COE = θ. In ΔOEC, OE = O $\cos\theta$ = R $\cos\theta$.

$$OF = \sqrt{OQ^2 - FQ^2} = \sqrt{R^2 - L^2} \quad \text{and}$$

$CD = 2EC = 2R\sin\theta$

Therefore, area A of the rectangle

$ABCD = CD \times EF = 2R\sin\theta \times (OE - OF)$

$$= 2R\sin\theta\left(R\cos\theta - \sqrt{R^2 - L^2}\right)$$

$$\frac{dA}{d\theta} = 2R\left[\cos\theta\left(R\cos\theta - \sqrt{R^2 - L^2}\right) - R\sin^2\theta\right]$$

$$\frac{dA}{d\theta} = 0 \Rightarrow R\cos 2\theta = \cos\theta\sqrt{R^2 - L^2}$$

$$\Rightarrow \quad 2R\cos^2\theta - \cos\theta\sqrt{R^2 - L^2} - R = 0$$

$$\Rightarrow \quad \cos\theta = \frac{\sqrt{R^2 - L^2} \pm \sqrt{R^2 - L^2 + 8R^2}}{4R}$$

Neglecting the negative sign, we have

$$\cos\theta = \frac{\sqrt{R^2 - L^2} + \sqrt{9R^2 - L^2}}{4R}$$

Also it is easy to that for this value of θ, $\left(\dfrac{d^2A}{d\theta^2}\right) < 0$. So A is maximum for this value of θ.

$$AB = CD = 2R\sin\theta = 2R\sqrt{1-\frac{1}{16R^2}\left\{\sqrt{R^2-L^2}+\sqrt{9R^2-L^2}\right\}^2}$$

$$=\frac{1}{2}\sqrt{6R^2+2L^2-2\sqrt{\left(R^2-L^2\right)\left(9R^2-L^2\right)}}$$

$$BC = R\cos\theta-\sqrt{R^2-L^2}$$

$$=\frac{\sqrt{R^2-L^2}+\sqrt{9R^2-L^2}-4\sqrt{R^2-L^2}}{4}.$$

Q.8. *Show that the function f defined by*

$$f(x)=\lim_{t\to\infty}\frac{(1+\sin\pi x)^t-1}{(1+\sin\pi x)^t+1}$$

is discontinuous at any $x \in N \cup \{0\}$.

Sol. Since $\pi x > 0$, if $2m\pi < \pi x < (2m + 1)\pi$,

$\sin\pi x < 0$ if $(2m + 1)\pi < \pi x < (2m + 2)\pi$

and $\sin \pi x = 0$ if $x = 0, 1, 2........$

$$f(x)=\begin{cases}\lim\limits_{t\to\infty}\dfrac{1-\dfrac{1}{(1+\sin\pi x)^t}}{1+\dfrac{1}{(1+\sin\pi x)^t}} & \text{if } 2m<x<2m+1\\[2ex] \lim\limits_{t\to\infty}\dfrac{(1+\sin\pi x)^t-1}{(1+\sin\pi x)^t+1} & \text{if } (2m+1)<x<2m+2\\[2ex] 0 & \text{if } x=0,1,2........\end{cases}$$

$$=\begin{cases}1 & 2m<x<2m+1\\ -1 & (2m+1)<x<2m+2\\ 0 & x=0,\ 1,\ 2\ ....\end{cases}$$

If $x \in N \cup \{0\}$, $f(x) = 0$ but $\lim\limits_{x\to 0} f(x)=0$ or -1 according as

$x \in (2m, 2m + 1)$ or $x \in (2m + 1, 2m + 2)$.

Q.9. *Show that if a, b, c and d are real numbers and $ac = 2(b + d)$, then at least one of the equations $x^2 + ax + b = 0$, $x^2 + cx + d = 0$ has real roots.*

Sol. The discriminant D_1 of equation $x^2 + ax + b = 0$ is $D_1 = a^2 - 4b$ and the discriminant D_2 of equations $x^2 + cx + d = 0$ is $D_2 = c^2 - 4d$. If both the equations $x^2 + ax + b = 0$ and $x^2 + cx + d = 0$ have non-real complex roots, then $D_1 < 0$, $D_2 < 0$. Therefore $D_1 + D_2 < 0$, or

$$a^2 + c^2 - 4(b + d) < 0 \qquad ...(i)$$

As $ac = 2(b + d)$, therefore, $a^2 + c^2 - 2ac < 0$ or $(a - c)^2 < 0$.

Since a, b, c, d are real, $(a - c)^2 \not< 0$. Therefore, our assumption that both the given equations have non-real complex roots must be wrong. Hence, at least one of $x^2 + ax + b = 0$ and $x + cx + d = 0$ has real roots.

Q.10. *An unbiased die is thrown. If a multiple of 3 appears, two balls are drawn from box A. If a multiple of 3 does not appear, two balls are drawn from box B. The balls drawn are found to be of different colours. Box A contains 3 white, 2 black balls and Box B contains 4 white and 1 black ball. Find the probability that the balls were drawn from box B if the balls are drawn with replacement.*

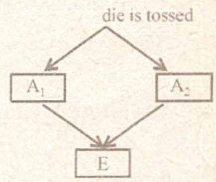
Sol. A_1 : event that balls were drawn from box A.

A_2 : event that balls were drawn from box B.

E : balls are of different colours.

$P(A_1)$ = P(balls from A) = P (3, 6 on die) $=\frac{2}{6}=\frac{1}{3}$

$P(A_2)$ = P(balls from B) = P(1, 2, 4, 5 on die) $=\frac{4}{6}=\frac{2}{3}$



$P(E/A_1)$ = P(1W, 1B from box A)

= P(1 success in two trials)

(taking W balls as success)

$$= {}^2C_1pq = 2\left(\frac{3}{5}\right)\left(\frac{2}{5}\right)=\frac{12}{25}$$

$P(E/A_2)$ = P(1W, 1B from box B)

= P(1 success in two trials) (taking W balls as success)

$$= {}^2C_1pq = 2\left(\frac{4}{5}\right)\left(\frac{1}{5}\right)=\frac{8}{25}$$

$$P(E) = P(A_1)\,.\,P\left(\frac{E}{A_1}\right)+P(A_2).P\left(\frac{E}{A_2}\right)$$

$$=\frac{1}{3}\left(\frac{12}{25}\right)+\frac{2}{3}\left(\frac{8}{25}\right)=\frac{28}{75}$$

Required probability is

$$P(A_2/E)=\frac{P(A_2)\,P(E/A_2)}{P(E)}=\frac{\frac{2}{3}\left(\frac{8}{25}\right)}{\frac{28}{75}}=\frac{4}{7}.$$

IIT-MAIN CHALLENGER-5 SOLUTIONS

(Contd. from Page 58)

Now PF is clearly equal to CL, *i.e.*, perpendicular from C (0, 0) on (*i*).

$$\therefore \quad PF = CL = \frac{1}{\sqrt{\left(\frac{\sec^2\phi}{a^2}+\frac{\tan^2\phi}{b^2}\right)}}$$

$$=\frac{ab}{\sqrt{\left(b^2\sec^2\phi+a^2\tan^2\phi\right)}}=\frac{ab}{(a/b).PG} \qquad \text{[From (iii)]}$$

$\therefore$ PG . PF = $b^2 = (CB)^2$

and $$PF=\frac{ab}{\sqrt{\left(b^2\sec^2\phi+a^2\tan^2\phi\right)}}=\frac{ab}{(b/a).Pg} \qquad \text{[From (iv)]}$$

PF. P$g = a^2 = (CA)^2$.

MODEL SOLVED PAPER

BITSAT-2005

(Online Exam to be held during 10 April-20 June, 2005)

1. Locus of complex number z such that $\log_{1/2}\frac{|z-1|+3}{5|z-1|-10}<2$ is given by

(a) $|z-1|<22$ (b) $|z-1|>2$

(c) $2<|z-1|<22$ (d) None of these.

2. The value of a for which the equation

$$(a^2+a-6)x^2+(a^2-4)x+(a^2-2a)=0$$

will have three solutions or more solutions is

(a) 1 (b) 0

(c) 2

(d) Cannot have three or more solutions.

3. The real values of a for which the equation

$$ax^2-(a+1)x+3=0$$

have roots lying between 1 and 2.

(a) $a<1$ (b) $a>\frac{1}{3}$

(c) $\frac{1}{3}<a<1$ (d) No value.

4. If a, b, c be the pth, qth, rth terms respectively of an AP and GP both, then product of the roots of the equation $a^b b^c c^a x^2 - abcx + a^c b^a c^b = 0$ is equal to

(a) -1 (b) 1

(c) 2 (d) $(b-c)(c-a)(a-b)$

5. If $\frac{\log x}{y-z}=\frac{\log y}{z-x}=\frac{\log z}{x-y}$, then minimum value of $x^x+y^y+z^z$ is

(a) 0 (b) 1

(c) 3 (d) Does not exist.

6. $\frac{1+3}{1!}\log_e 3+\frac{1+3^2}{2!}(\log_e 3)^2+\frac{1+3^3}{3!}(\log_e 3)^3+\ldots\ldots\infty=$

(a) 28 (b) 30

(c) e^3 (d) None of these.

7. Two numbers are chosen from 1, 3, 5, 7, ..., 147, 149, 151 and multiplied together in all possible ways. The number of ways which will give us the product a multiple of 5 is

(a) 74 (b) 75

(c) 76 (d) None of these.

8. The value of $C_0^2+\frac{C_1^2}{2}+\frac{C_2^2}{3}+\ldots\ldots+\frac{C_n^2}{n+1}$, where C_0, C_1, ..., C_n are binomial coefficients in the expansion of $(1+x)^n$, is

(a) 0 (b) ${}^{2n}C_n$

(c) $\frac{{}^{2n+1}C_{n+1}}{n+1}$ (d) ${}^{2n+1}C_{n+1}$

9. Let x, y, z (>0) are respectively the 2nd, 3rd and 4th terms of a G.P. and

$$\Delta=\begin{vmatrix} x^k & x^{k+1} & x^{k+2} \\ y^k & y^{k+1} & y^{k+2} \\ z^k & z^{k+1} & z^{k+2}\end{vmatrix}=(r-1)^2\left(1-\frac{1}{r^2}\right),$$

where r is the common ratio. Then $k=$

(a) -1 (b) 1

(c) 0 (d) None of these.

10. x, y, u, v are four real numbers such that

$x+y=u+v$ and $x^2+y^2=u^2+v^2$. Then $x^{100}+y^{100}=$

(a) $(x+y)^{100}-(u+v)^{100}$ (b) $u^{100}+v^{100}$

(c) $u^{101}+v^{101}$ (d) None of these.

11. Let $f(x)=\frac{2x}{2x^2+5x+2}$ and $g(x)=\frac{1}{x+1}$. Then the set of real values of x for which $f(x)>g(x)$ is

(a) $(-2,-1)\cup\left(-\frac{2}{3},-\frac{1}{2}\right)$ (b) $(-2,-1)$

(c) $\left(-\frac{2}{3},-\frac{1}{2}\right)$

(d) $(-\infty,-2)\cup\left(-1,-\frac{2}{3}\right)\cup\left(-\frac{1}{2},\infty\right)$

12. If $\frac{x}{\tan(\theta+\alpha)}=\frac{y}{\tan(\theta+\beta)}=\frac{z}{\tan(\theta+\gamma)}$, then $\sum\frac{x+y}{x-y}\sin^2(\alpha-\beta)=$

(a) xyz (b) $xy+yz+zx$

(c) 0 (d) None of these.

13. The equation $2\cos x+\cos 2kx=3$ has only one solution, then k can have

(a) any integral value. (b) any rational value.

(c) any irrational value. (d) any even integral value.

14. If $\cot^{-1}x+\cot^{-1}y+\cot^{-1}z=\frac{\pi}{2}$, then $x+y+z$ equals

(a) xyz (b) $xy+yz+zx$

(c) $2xyz$ (d) None of these.

15. If the tangents of the $\angle$s of a Δ are in A.P., then the squares of the sides are in the ratio

(a) $x^2 : x^2+3 : x^2+6$

(b) $x^2(x^2+9) : (x^2+3)^2 : 9(x^2+1)$

(c) $x^2 : x^2(x^2+3) : x^2(x^2+9)$

(d) None of these.

16. AB is a vertical pole. The end A is on the level ground. C is the middle point of AB. P is a point, on the